AIPS Memo 27

Non-linear Coordinate Systems in $\mathcal{AIPS}$
Reissue of November 1983 version

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Abstract

$\mathcal{AIPS}$ has been revised recently to support several projective geometries and a non-linear velocity axis. The present memorandum contains a description of the FITS-like nomenclature used to describe these coordinates and of the algebra implemented to compute their values. The use of Galactic as well as Celestial coordinates is explicated. A guide to the routines in $\mathcal{AIPS}$ which implement these constructs is given.

Added January 1993: One typographical correction of the old memo and one substantial difference between this memo and the 1993 proposed FITS conventions are footnoted where appropriate.

1 Introduction

A variety of non-linear coordinate systems are in widespread use in astronomy. Both $\mathcal{AIPS}$ and the FITS standard, on which the $\mathcal{AIPS}$ format is based, suffer from considerable ambiguity in the description of such coordinates. I have developed a self-consistent method to label such coordinates and have implemented some of them in $\mathcal{AIPS}$. This memorandum describes the algebra, the header parameters, and the commons used in this implementation.

2 Velocity and Frequency

Velocity and frequency are frequently used as approximately equivalent axes. The $\mathcal{AIPS}$ verb `ALTDEF` allows the user to provide the parameters for an alternative velocity definition of the frequency axis. The verb `ALTSWTCH` switches the main header and alternative axis descriptions. $\mathcal{AIPS}$ currently supports three axis types of this kind: '`FREQ....`' regularly gridded in frequency, '`VELO....`' regularly gridded in velocity, and '`FELO...`' regularly gridded in frequency but expressed in velocity units in the optical convention. The reverse of '`FELO....`' has not been implemented since I do not expect it to arise. The observed velocity, V, is the sum of the projected velocity of the observer with respect to some inertial system, $V_{\rm OBS}$, and the projected velocity of the astronomical object with respect to that system, $V_{\rm S}$. The "`....`" above are four characters which document the choice of inertial system. Currently '`-LSR`', '`-HEL`', and '`-OBS`' are implemented for Local Standard of Rest, heliocentric, and geocentric systems, but more codes could be added easily.

Since astronomical velocities are sometimes large, we should use a proper set of relativistic formulæ. For *true* velocities, denoted by lower case letters, the relativistic sum of two velocities is

$$v = \frac{v_{\rm S} + v_{\rm OBS}}{1 + v_{\rm S} v_{\rm OBS}/c^2}\,, \tag{1}$$

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while the Doppler shift is given by

$$\nu' = \nu \left(\frac{c-v}{c+v}\right)^{1/2} . \tag{2}$$

For some reason, astronomers do not normally use a true velocity. Instead, there are two conventions used to express the relationship of frequency and velocity: the "optical" and the "radio". In the radio convention,

$$V = -c\,(\nu' - \nu_0)/\nu_0$$

or

$$\frac{V}{c} = 1 - \left(\frac{c-v}{c+v}\right)^{1/2} , \tag{3}$$

which reverses to

$$\frac{v}{c} = \frac{2cV + V^2}{2c^2 - 2cV + V^2} . \tag{4}$$

Substituting equation 1 in equation 3 and replacing $v_{\rm S}$ and $v_{\rm OBS}$ with appropriate versions of equation 4, we obtain after a lot of manipulation,

$$V = V_{\rm S} + V_{\rm OBS} - V_{\rm S} V_{\rm OBS}/c .$$

But we have observed at regular frequency spacings in our rest frame, so

$$V = -\frac{c}{\nu_0}\left(\nu_R + \delta_\nu (N - N_\nu) - \nu_0\right) ,$$

where c is the speed of light, ν_0 the rest frequency, ν_R the reference frequency, δ_ν the increment in frequency per pixel, N the pixel, and N_ν the frequency reference pixel. If the velocity of the object with respect to the reference frame V_R is given for a velocity reference pixel $N_{\rm V}$, then

$$\begin{aligned} V' &= V_{\rm OBS} + V_R - V_{\rm OBS} V_R/c \\ &= -\frac{c}{\nu_0}\left(\nu_R + \delta_\nu (N_{\rm V} - N_\nu) - \nu_0\right) . \end{aligned}$$

Noting that

$$\begin{aligned} V_{\rm S} &= \frac{V - V_{\rm OBS}}{1 - V_{\rm OBS}/c} \\ V_{\rm OBS} &= c\frac{V' - V_R}{c - V_R} \end{aligned}$$

and, doing a lot of substitutions and manipulations, we find that

$$V_{\rm S} = V_R + \delta_{\rm V}\,(N - N_{\rm V}) ,$$

where

$$\begin{aligned} \delta_{\rm V} &\equiv -\delta_\nu\,(c - V_R)\,/\nu_x \\ \nu_x &\equiv \nu_R + \delta_\nu\,(N_{\rm V} - N_\nu) . \end{aligned}$$

In the optical convention, the definition of velocity is

$$\begin{aligned} \frac{V}{c} &= -\frac{\nu' - \nu_0}{\nu'} \\ &= \left(\frac{c+v}{c-v}\right)^{1/2} - 1 \end{aligned}$$

which reverses to

$$\frac{v}{c} = \frac{2cV + V^2}{2c^2 + 2cV + V^2}\,.$$

Doing a similar set of substitutions, we derive

$$V = V_{\rm S} + V_{\rm OBS} + V_{\rm S} V_{\rm OBS}/c\,.$$

Manipulating the velocity equations to eliminate $V_{\rm OBS}$, we obtain

$$V_{\rm S} = V_R + \frac{(c + V_R)(V - V\,')}{c + V\,'}$$

or, substituting the frequency information,

$$V_{\rm S} = V_R - \frac{\delta_\nu (c + V_R)(N - N_{\rm V})}{\nu_R + \delta_\nu (N - N_\nu)}\,.$$

For header purposes, the velocity increment is the slope of $V_{\rm S}$ at $N_{\rm V}$

$$\delta_{\rm V} = -\delta_\nu (c + V_R)/\nu_x$$

and, for coordinate computations,

$$V_{\rm S} = V_R + \frac{\delta_{\rm V}(N - N_{\rm V})}{1 + (N - N_{\rm V})\,\delta_\nu/\nu_x}\,.$$

The *AIPS* catalog header provides storage locations for the current axis description:

`CAT8(K8CRV+J)`	reference pixel value, ν_R or V_R
`CAT4(K4CIC+J)`	increment at reference pixel, δ_ν or $\delta_{\rm V}$
`CAT4(K4CRP+J)`	reference pixel location, N_ν or $N_{\rm V}$
`CAT4(K4CTP+2*J)`	axis label

where the frequency or velocity is the $(J+1)^{\rm st}$ axis. The alternative reference information is stored in

`CAT8(K8RST)`	rest frequency, ν_0
`CAT4(K4ARP)`	alternate reference pixel, $N_{\rm V}$ or N_ν
`CAT8(K8ARV)`	alternate reference value: either V_R or ν_x
`CAT2(K2ALT)`	axis type code: 1 LSR, 2 HEL, 3 OBS plus 256 if radio, 0 implies no alternate axis

Note that ν_x (not ν_R) is stored when velocity is in the main axis description. This allows *AIPS* to recover the frequency increment. For coordinate computations, the routine `SETLOC` (to be described in more detail later) prepares the variable `AXDENU` in the common `/LOCATI/`, where `AXDENU` $\equiv \delta_\nu/\nu_x = -\delta_{\rm V}/(c + V_R)$. This parameter is, of course, only used for axes labeled '`FELO....`'.

3 Projective Coordinate Systems

There are three projections to the tangent plane illustrated in Figure 1 which are in frequent use in astronomy. The TAN projection is common in optical astronomy and the SIN projection is common in radio aperture synthesis. The ARC projection, in which angular distances are preserved, is used in Schmidt telescopes (to first order) and is also used in mapping with single dish radio telescopes. Another geometry, used by the WSRT, involves a projection to a plane perpendicular to the North Celestial Pole. *AIPS* now supports all four of these geometries with full non-linear computations of the coordinate values. The choice of geometry is conveyed in the last four characters of the axis type as '`....-TAN`', '`....-SIN`', '`....-ARC`', and '`....-NCP`'. The kind of coordinate is conveyed in the first four characters as '`RA--....`', '`DEC-....`',

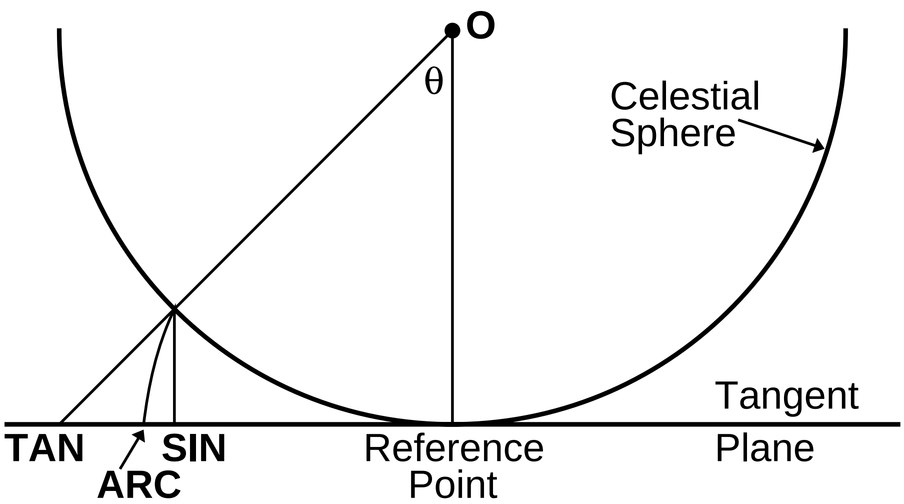

Figure 1: Three projections to the tangent plane

'`GLON....`', '`GLAT....`', '`ELON....`', and '`ELAT....`' for longitude and latitude in the Celestial, Galactic, and Ecliptic systems.

In a projected plane, the position of a point (x, y) with respect to the coordinate reference point in an arbitrary linear system may be represented as

$$\begin{aligned} x &= L\cos\rho + M\sin\rho \\ y &= M\cos\rho - L\sin\rho\,, \end{aligned} \tag{5}$$

where ρ is a rotation, L is the direction cosine parallel to latitude at the reference pixel, and M is the direction cosine parallel to longitude at the reference pixel. Both the (x, y) and (L, M) systems are simple linear, perpendicular systems. If we represent longitude and latitude with the symbols α and δ, the fun arises in solving the four problems: (i) given α, δ find x, y; (ii) given x, y find α, δ; (iii) given x, δ find α, y; and (iv) given α, y find x, δ. I will derive the answers, used by $\mathcal{AIPS}$, to these problems in the remainder of this section. The spherical coordinates are defined in Figure 2. Using $\Delta\alpha \equiv \alpha - \alpha_0$, the usual spherical triangle rules provide the basic formulæ

$$\cos\theta = \sin\delta\sin\delta_0 + \cos\delta\cos\delta_0\cos\Delta\alpha \tag{6}$$

$$\sin\theta\sin\phi = \cos\delta\sin\Delta\alpha \tag{7}$$

$$\sin\theta\cos\phi = \sin\delta\cos\delta_0 - \cos\delta\sin\delta_0\cos\Delta\alpha\,. \tag{8}$$

3.1 TAN geometry

3.1.1 Find x, y from α, δ

$$\begin{aligned} L &= \tan\theta\sin\phi \\ M &= \tan\theta\cos\phi \end{aligned}$$

or

$$\begin{aligned} L &= \frac{\cos\delta\sin\Delta\alpha}{\sin\delta\sin\delta_0 + \cos\delta\cos\delta_0\cos\Delta\alpha} \\ M &= \frac{\sin\delta\cos\delta_0 - \cos\delta\sin\delta_0\cos\Delta\alpha}{\sin\delta\sin\delta_0 + \cos\delta\cos\delta_0\cos\Delta\alpha}\,. \end{aligned} \tag{9}$$

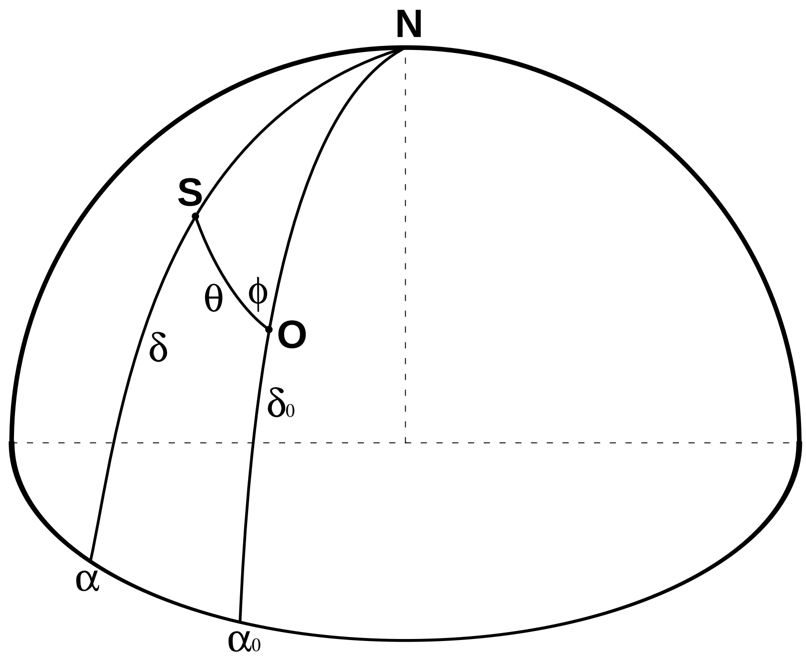

Figure 2: Celestial coordinates of reference position O and source S

x and y may then be determined by equations 5.

3.1.2 Find α, δ from x, y

L, M may be found by equations 5. Then, equations 9 may be inverted to yield

$$\begin{aligned}\cos\Delta\alpha &= \tan\delta\,\frac{\cos\delta_0 - M\sin\delta_0}{M cos\delta_0 + \sin\delta_0}\\ \sin\Delta\alpha &= \tan\delta\,\frac{L}{M\cos\delta_0 + \sin\delta_0}\,.\end{aligned}$$

Then

$$\begin{aligned}\alpha &= \alpha_0 + \tan^{-1}\left(\frac{L}{\cos\delta_0 - M\sin\delta_0}\right)\\ \delta &= \tan^{-1}\left(\cos\Delta\alpha\,\frac{M\cos\delta_0 + \sin\delta_0}{\cos\delta_0 - M\sin\delta_0}\right)\,.\end{aligned}$$

3.1.3 Find α, y from x, δ

Equations 5 and 9 may be combined to yield

$$\cos\rho\sin\Delta\alpha - (x\cos\delta_0 + \sin\delta_0\sin\rho)\cos\Delta\alpha = x\tan\delta\sin\delta_0 - \tan\delta\cos\delta_0\sin\rho\,.$$

Thus, if

$$\begin{aligned}A &\equiv \cos\rho\\ B &\equiv x\cos\delta_0 + \sin\delta_0\sin\rho\,,\end{aligned}$$

then

$$\begin{aligned}
\alpha &= \alpha_0 + \tan^{-1}\left(\frac{B}{A}\right) + \sin^{-1}\left(\frac{\tan\delta\,(x\sin\delta_0 - \cos\delta_0\sin\rho)}{\sqrt{A^2+B^2}}\right) \\
y &= \frac{\sin\delta\cos\delta_0\cos\rho - \cos\delta\sin\delta_0\cos\Delta\alpha\cos\rho - \cos\delta\sin\Delta\alpha\sin\rho}{\sin\delta\sin\delta_0 + \cos\delta\cos\delta_0\cos\Delta\alpha}\,.
\end{aligned}$$

3.1.4 Find x, δ from x, y

The above equation for y may be inverted to yield

$$\delta = \tan^{-1}\left(\frac{\sin\delta_0\cos\Delta\alpha\cos\rho + \sin\Delta\alpha\sin\rho + y\cos\delta_0\cos\Delta\alpha}{\cos\delta_0\cos\rho - y\sin\delta_0}\right)\,.$$

Then,

$$x = \frac{\sin\Delta\alpha\cos\rho + \tan\delta\cos\delta_0\sin\rho - \sin\delta_0\cos\Delta\alpha\sin\rho}{\tan\delta\sin\delta_o + \cos\delta_0\cos\Delta\alpha}\,.$$

3.2 SIN geometry

3.2.1 Find x, y from α, δ

$$\begin{aligned}
L &= \sin\theta\sin\phi \\
M &= \sin\theta\cos\phi\,,^{1}
\end{aligned}$$

or, using equations 7 and 8,

$$\begin{aligned}
L &= \cos\delta\sin\Delta\alpha \\
M &= \sin\delta\cos\delta_0 - \cos\delta\sin\delta_0\cos\Delta\alpha\,.
\end{aligned} \tag{10}$$

Then x and y may be determined by equations 5.

3.2.2 Find α, δ from x, y

Equations 5 may be reversed to find L and M from x and y. Using equation 10,

$$\cos^2\delta\cos^2\Delta\alpha = \cos^2\delta - L^2 = (\sin\delta\cos\delta_0 - M)^2/\sin^2\delta_0$$

or, with trigonometric substitutions,

$$\begin{aligned}
\delta &= \sin^{-1}\left(M\cos\delta_0 + \sin\delta_0\sqrt{1-L^2-M^2}\right) \\
\alpha &= \alpha_0 + \tan^{-1}\left(\frac{L}{\cos\delta_0\sqrt{1-L^2-M^2} - M\sin\delta_0}\right)\,.
\end{aligned}$$

3.2.3 Find α, y from x, δ

Equations 5 and 10 may be combined to yield

$$(\cos\delta\cos\rho)\sin\Delta\alpha - (\cos\delta\sin\delta_0\sin\rho)\cos\Delta\alpha = x - \sin\delta\cos\delta_0\sin\rho\,.$$

[1] **1993:** The original Memo had a typo on this equation, declaring $M = \sin\theta\cos\theta$.

Thus, if

$$\begin{aligned} A &\equiv \cos\rho \\ B &\equiv \sin\delta_0 \sin\rho\,, \end{aligned}$$

then,

$$\alpha = \alpha_0 + \tan^{-1}\left(\frac{B}{A}\right) + \sin^{-1}\left(\frac{x - \sin\delta\cos\delta_0\sin\rho}{\cos\delta\sqrt{A^2+B^2}}\right)$$

and

$$\begin{aligned} y &= M\cos\rho - L\sin\rho \\ &= \sin\delta\cos\delta_0\cos\rho - \cos\delta\sin\delta_0\cos\Delta\alpha\cos\rho - \cos\delta\sin\Delta\alpha\sin\rho\,. \end{aligned}$$

3.2.4 Find x, δ from α, y

From the above equation

$$y = (\cos\delta_0\cos\rho)\sin\delta - (\sin\delta_0\cos\Delta\alpha\cos\rho + \sin\Delta\alpha\sin\rho)\cos\delta\,.$$

Thus, if

$$\begin{aligned} A &\equiv \cos\delta_0\cos\rho \\ B &\equiv \sin\delta_0\cos\Delta\alpha\cos\rho + \sin\Delta\alpha\sin\rho\,, \end{aligned}$$

then

$$\begin{aligned} \delta &= \tan^{-1}\left(\frac{B}{A}\right) + \sin^{-1}\left(\frac{y}{\sqrt{A^2+B^2}}\right) \\ x &= \cos\delta\sin\Delta\alpha\cos\rho + \sin\delta\cos\delta_0\sin\rho - \cos\delta\sin\delta_0\cos\Delta\alpha\sin\rho\,. \end{aligned}$$

3.3 ARC geometry

3.3.1 Find x, y from α, δ

$$\begin{aligned} L &= \theta\sin\phi \\ M &= \theta\cos\phi \end{aligned} \tag{11}$$

or, using equations 7, 8, and 6,

$$\begin{aligned} L &= \left(\frac{\theta}{\sin\theta}\right)\cos\delta\sin\Delta\alpha \\ M &= \left(\frac{\theta}{\sin\theta}\right)(\sin\delta\cos\delta_0 - \cos\delta\sin\delta_0\cos\Delta\alpha) \\ \theta &= \cos^{-1}(\sin\delta\sin\delta_0 + \cos\delta\cos\delta_0\cos\Delta\alpha)\,. \end{aligned} \tag{12}$$

We note that the sign of θ, ambiguous in an $\cos^{-1}$, is irrelevant here because it is used only in the form $\theta/\sin\theta$. Equations 5 are then used to find x and y.

3.3.2 Find α, δ from x, y

Equations 5 are reversed to determine L and M from x and y. Then equations 11 yield

$$|\theta| = \sqrt{L^2 + M^2}$$

directly, while equations 12 for M and $\cos\theta$, give

$$\delta = \sin^{-1}\left(\frac{M\cos\delta_0}{\theta/\sin\theta} + \sin\delta_0\cos\theta\right)$$

and equations 12 for L yields

$$\alpha = \alpha_0 + \sin^{-1}\left(\frac{\sin\theta}{\theta}\,\frac{L}{\cos\delta}\right).$$

Since these are fairly simple exact expressions, I do not see the need to use approximations as is normally done in the literature on the Schmidt geometry. Since $\theta/\sin\theta$ is not susceptible to trigonometric identities, the "cross-product" problems (below) do not have exact solutions. Instead, $\mathcal{AIPS}$ implements iterative methods.

3.3.3 Find α, y from x, δ

From the previous section

$$\begin{aligned}
\sin\delta &= M\cos\delta_0\,\frac{\sin\theta}{\theta} + \sin\delta_0\cos\theta \\
&= \frac{\sin\theta}{\theta}(x\sin\rho + y\cos\rho)\cos\delta_0 + \cos\theta\sin\delta_0
\end{aligned}$$

or

$$y = \frac{\sin\delta - \sin\delta_0\cos\theta - x\sin\rho\cos\delta_0\,(\sin\theta/\theta)}{\cos\rho\cos\delta_0\,(\sin\theta/\theta)}.$$

This can be solved iteratively. We begin by setting y to 0 and compute

$$\theta = \sqrt{x^2+y^2}$$

followed by the above formula for y. Then we improve the estimate of θ and repeat. When convergence is achieved, we may then compute

$$\alpha = \alpha_0 + \sin^{-1}\left(\frac{\sin\theta}{\theta}\,\frac{x\cos\rho - y\sin\rho}{\cos\delta}\right).$$

3.3.4 Find x, δ from α, y

This problem also requires an iterative method which is a bit messier. In order to restrict the main computations to terms involving uncertainties which are no worse than second order in x, an $\cos^{-1}$ is required. The method begins by setting $x = 0$. Then

$$\begin{aligned}
\theta &= \sqrt{x^2+y^2} \\
\delta &= \tan^{-1}\left(\frac{\tan\delta_0}{\cos\Delta\alpha}\right) + \mathrm{sign}(y)\cos^{-1}\left(\frac{\cos\theta}{\sqrt{1-\cos^2\delta_0\sin^2\Delta\alpha}}\right) \\
x &= \frac{y\sin\rho + \cos\delta\sin\Delta\alpha\,(\theta/\sin\theta)}{\cos\rho},
\end{aligned}$$

where

$$\mathrm{sign}(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$$

and repeat. The second equation above is a rewritten version of equation 6 and the third equation is equations 12 for L rewritten in terms of x and y.

3.4 NCP geometry[2]

3.4.1 Find x, y from α, δ

From *Data Processing for the Westerbork Synthesis Radio Telescope*, W. N. Brouw (1971), we have

$$\begin{aligned} L &= \cos\delta \sin\Delta\alpha \\ M &= (\cos\delta_0 - \cos\delta\cos\Delta\alpha)/\sin\delta_0 \, . \end{aligned} \tag{13}$$

Equations 5 then provide x and y.

3.4.2 Find α, δ from x, y

The reverse of equations 5 yield L and M from x and y. Then, combining equations 13,

$$\begin{aligned} \alpha &= \alpha_0 + \tan^{-1}\left(\frac{L}{\cos\delta_0 - M\sin\delta_0}\right) \\ \delta &= \mathrm{sign}(\delta_0)\,\cos^{-1}\left(\frac{\cos\delta_0 - M\sin\delta_0}{\cos\Delta\alpha}\right) . \end{aligned}$$

3.4.3 Find α, y from x, δ

Since

$$x = L\cos\rho + M\sin\rho$$

we use equations 13 and rearrange to obtain

$$(\cos\delta\cos\rho)\sin\Delta\alpha - \left(\frac{\cos\delta\sin\rho}{\sin\delta_0}\right)\cos\Delta\alpha = \left(\frac{x - \cos\delta_0\sin\rho}{\sin\delta_0}\right) .$$

Thus, if

$$\begin{aligned} A &\equiv \cos\rho \\ B &\equiv \sin\rho/\sin\delta_0 \, , \end{aligned}$$

then

$$\begin{aligned} \alpha &= \alpha_0 + \tan^{-1}\left(\frac{B}{A}\right) + \sin^{-1}\left(\frac{x\sin\delta_0 - \cos\delta_0\sin\rho}{\cos\delta\sin\delta_0\sqrt{A^2+B^2}}\right) \\ y &= \cos\rho\left(\frac{\cos\delta_0 - \cos\delta\cos\Delta\alpha}{\sin\delta_0}\right) - \sin\rho\cos\delta\sin\Delta\alpha \, . \end{aligned}$$

3.4.4 Find x, δ from α, y

Substituting equations 13 in equation 5 for y and rearranging,

$$\delta = \mathrm{sign}(\delta_0)\,\cos^{-1}\left(\frac{\cos\delta_0\cos\rho - y\sin\delta_0}{\cos\Delta\alpha\cos\rho + \sin\Delta\alpha\sin\rho\sin\delta_0}\right) .$$

Then

$$\begin{aligned} x &= \cos\rho\cos\delta\sin\Delta\alpha + \sin\rho\left(\frac{\cos\delta_0 - \cos\delta\cos\Delta\alpha}{\sin\delta_0}\right) \\ &= \frac{\cos\delta_0\sin\Delta\alpha - y\,(\sin\Delta\alpha\sin\delta_0\cos\rho - \cos\Delta\alpha\sin\rho)}{\cos\Delta\alpha\cos\rho + \sin\alpha\Delta\sin\rho\sin\delta_0} . \end{aligned}$$

[2] **1993:** This "geometry" is actually a `SIN` geometry with the North Celestial Pole as the tangent point; it is, therefore, deprecated in the 1993 proposal.

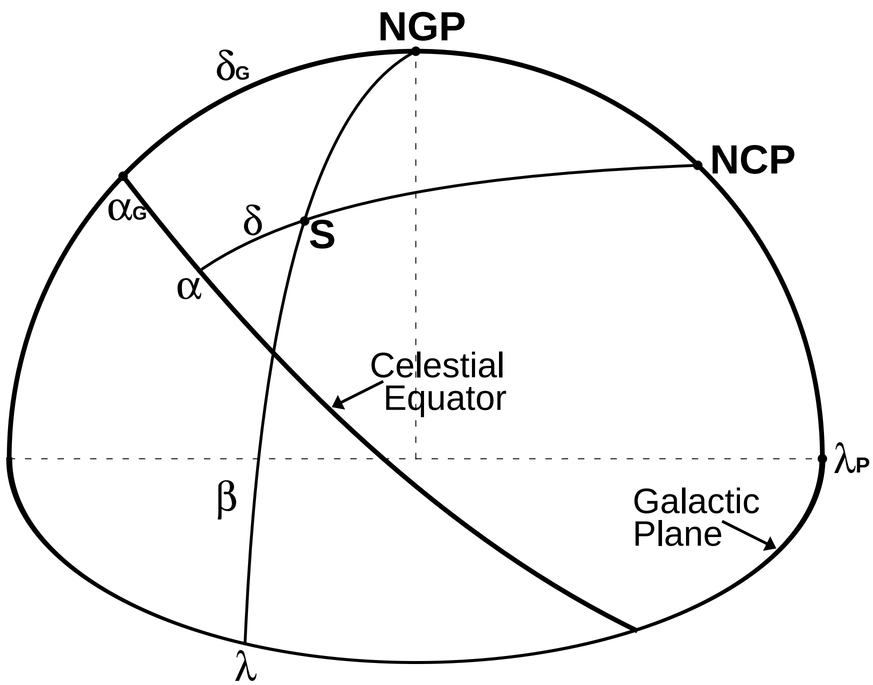

Figure 3: Celestial and Galactic coordinates of source S

4 Galactic Coordinates

For observers of Galactic objects, it is often more relevant to display their images in Galactic rather than Celestial coordinates. It may be shown that for any geometric projection to the tangent plane, the two systems are equivalent except for a change in the reference coordinates α_0, δ_0 and the rotation angle ρ. Note that the NCP geometry does not use a tangent plane and, hence, cannot be in alternate coordinates. A verb, `CELGAL`, has been implemented in *AIPS* to convert the catalog header from Celestial to Galactic coordinates and back again. The mathematics used in this conversion is derived below and illustrated on the Celestial sphere in Figure 3.

Let $\alpha_{\mathrm{G}}, \delta_{\mathrm{G}}$ be the Celestial coordinates of the North Galactic pole (192.25, 27.4 degrees) and λ_P be the Galactic longitude of the North celestial pole (123.0 degrees). To do the conversion we need only solve the spherical triangle NGP–S–NCP:

$$\begin{aligned}
\sin\beta &= \sin\delta\sin\delta_{\mathrm{G}} + \cos\delta\cos\delta_{\mathrm{G}}\cos(\alpha-\alpha_{\mathrm{G}}) \\
\cos\beta\sin(\lambda_P-\lambda) &= \cos\delta\sin(\alpha-\alpha_{\mathrm{G}}) && (14) \\
\cos\beta\cos(\lambda_P-\lambda) &= \sin\delta\cos\delta_{\mathrm{G}} - \cos\delta\sin\delta_{\mathrm{G}}\cos(\alpha-\alpha_{\mathrm{G}}) && (15)
\end{aligned}$$

or

$$\lambda = \lambda_P + \tan^{-1}\left(\frac{\cos\delta\sin(\alpha-\alpha_{\mathrm{G}})}{\cos\delta\sin\delta_{\mathrm{G}}\cos(\alpha-\alpha_{\mathrm{G}}) - \sin\delta\cos\delta_{\mathrm{G}}}\right).$$

The reverse formulæ are equally straight forward:

$$\begin{aligned}
\alpha &= \alpha_{\mathrm{G}} + \tan^{-1}\left(\frac{\cos\beta\sin(\lambda-\lambda_P)}{\cos\beta\sin\delta_{\mathrm{G}}\cos(\lambda-\lambda_P) - \sin\beta\cos\delta_{\mathrm{G}}}\right) \\
\delta &= \sin^{-1}\left(\sin\beta\sin\delta_{\mathrm{G}} + \cos\beta\cos\delta_{\mathrm{G}}\cos(\lambda-\lambda_P)\right)
\end{aligned}$$

The proof that a rotation applies and the derivation of its value is messier. Using the SIN geometry, we evaluate

$$\begin{aligned} L' &\equiv \cos\beta \sin(\lambda - \lambda_0) \\ &= \cos\beta \sin(\lambda - \lambda_P)\cos(\lambda_0 - \lambda_P) - \cos\beta\cos(\lambda - \lambda_P)\sin(\lambda_0 - \lambda_P)\,. \end{aligned}$$

Substituting from equations 14, expanding, and using

$$\sin(\alpha - \alpha_{\mathrm{G}}) = \sin\Delta\alpha\cos(\alpha_0 - \alpha_{\mathrm{G}}) + \cos\Delta\alpha\sin(\alpha_0 - \alpha_{\mathrm{G}})\,,$$

we obtain

$$L' = L\left(\frac{\cos\delta_0\sin\delta_{\mathrm{G}} - \sin\delta_0\cos\delta_{\mathrm{G}}\cos(\alpha_0 - \alpha_{\mathrm{G}})}{\cos\beta_0}\right) + M\left(\frac{\cos\delta_{\mathrm{G}}\sin(\alpha_0 - \alpha_{\mathrm{G}})}{\cos\beta_0}\right)\,.$$

This is then a rotation R given by

$$R = \tan^{-1}\left(\frac{\cos\delta_{\mathrm{G}}\sin(\alpha_0 - \alpha_{\mathrm{G}})}{\cos\delta_0\sin\delta_{\mathrm{G}} - \sin\delta_0\cos\delta_{\mathrm{G}}\cos(\alpha_0 - \alpha_{\mathrm{G}})}\right)\,.$$

I have checked this using M and the TAN geometry and obtain the same result. The sign conventions are such that

$$\rho_{\mathrm{GAL}} = \rho_{\mathrm{CEL}} - R\,.$$

5 The *AIPS* Implementation

As in previous versions, positions are handled primarily through a "location" common named `/LOCATI/` and included via `DLOC.INC` and `CLOC.INC`. This common is initialized via a call to `SETLOC (IDEPTH)` where `IDEPTH` is a five-integer array giving the location of the current plane on axes 3 through 7. The image catalog header is required to be in common `/MAPHDR/`.

The contents of this common are used for the computation of positions and axis labeling. Some portions of the common have, however, wider potential uses. Four "primary" axes are identified in the common: the x-axis, the y-axis, and, where present, up to two of axes 3–7. The latter are "normally" used solely for labeling. However, when one of the x and y axes is a position axis (*e.g.*, RA, Glon) and the other is not, then the third primary axis is identified with the corresponding position axis (*e.g.*, Dec, Glat) and used in position computations. Such an axis is often called the "z" axis and occurs in transposed spectral line imagery among other places.

The parameters of the common are

RPVAL	R*8(4)	Reference pixel values
COND2R	R*8	Degrees to radians multiplier = $\pi/180$
AXDENU	R*8	δ_ν/ν_x when a FELO axis is present
RPLOC	R*4(4)	Reference pixel locations
AXINC	R*4(4)	Axis increments
CTYP	R*4(2,4)	Axis types
CPREF	R*4(2)	x, y axis prefixes for labeling
ROT	R*4	Rotation angle of position axes
SAXLAB	R*4(5,2)	Labels for axes 3 and 4 values (4 char/fp)
ZDEPTH	I*2(5)	Value of IDEPTH from SETLOC call
ZAXIS	I*2	1-relative axis number of z axis
AXTYP	I*2	Position axis code
CORTYP	I*2	Which position is which
LABTYP	I*2	Special x, y label request
SGNROT	I*2	Extra sign to apply to rotation
AXFUNC	I*2(7)	Kind of axis code
KLOCL	I*2	0-rel axis number–longitude axis
KLOCM	I*2	0-rel axis number–latitude axis
KLOCF	I*2	0-rel axis number–frequency axis
KLOCS	I*2	0-rel axis number–Stokes axis
KLOCA	I*2	0-rel axis number–"primary axis" 3
KLOCB	I*2	0-rel axis number–"primary axis" 4
NCHLAB	I*2(2)	Number of characters in SAXLAB

There are several sets of codes here which need additional explanation:

AXTYP	value = 0	no position-axis pair
	= 1	$x - y$ are position pair
	= 2	$x - z$ are position pair
	= 3	$y - z$ are position pair
	= 4	2 z axes form a pair
CORTYP	value = 0	linear x, y axes
	= 1	x is longitude, y is latitude
	= 2	y is longitude, x is latitude
	= 3	x is longitude, z is latitude
	= 4	z is longitude, x is latitude
	= 5	y is longitude, z is latitude
	= 6	z is longitude, y is latitude
LABTYP	value = $10 * ycode + xcode$	
	$.code$ = 0	use CPREF, CTYP
	= 1	use Ecliptic longitude
	= 2	use Ecliptic latitude
	= 3	use Galactic longitude
	= 4	use Galactic latitude
	= 5	use Right Ascension
	= 6	use Declination
AXFUNC	value = -1	no axis
	= 0	linear axis
	= 1	FELO axis
	= 2	SIN projection
	= 3	TAN projection
	= 4	ARC projection
	= 5	NCP projection

The KLOC. parameters have value -1 if the corresponding axis does not exist. If AXTYP is 2 or 3, the pointer

KLOCA will always point at the z axis. In this case, SETLOC does not have enough information to prepare SAXLAB(*,1). The string must be computed later when an appropriate x, y position is specified.

The four kinds of position computations are implemented in *AIPS* by the subroutines XYPIX, XYVAL, FNDX, and FNDY, respectively. These routines all use the location common, sort out the various combinations, deal with rotation where possible, and call lower level routines. The subroutines NEWPOS, DIRCOS, DIRRA, and DIRDEC actually implement the trigonometry of Section 3, but will seldom be of immediate interest to the general programmer. The character strings used to display axis values are prepared normally with the new subroutine AXSTRN. More general axis labeling problems are initialized with routines LABINI and SLBINI. The former calls SETLOC, prepares the z-axis string, and revises the axis description to match the user-requested labeling type. The latter calls LABINI and then deals with the special problems of slices. The subroutine AU7 implements the verbs ALTDEF, ALTSWTCH, and CELGAL.

Several new FITS keywords are now written and read by *AIPS*. They implement the new header parameters for the alternate axis description, the observed (pointing) position, and the coordinate shifts. Tentatively, these keywords are

VELREF	C*8	Velocity reference systems
ALTRVAL	R*8	Alternate reference value
ALTRPIX	R*8	Alternate reference pixel
RESTFREQ	R*8	Line rest frequency
OBSRA	R*8	Pointing position: RA of epoch
OBSDEC	R*8	Pointing position: DEC of epoch
XSHIFT	R*8	Sum of phase shifts: RA
YSHIFT	R*8	Sum of phase shifts: DEC.

These may change when a new FITS agreement is reached.

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