1 The Search Algorithm

1.1 Local rectangular coordinates

The angular coordinates Right Ascension (α) and Declination (δ) of a star specify a point $\hat{r}$[1] on the three dimensional unit celestial sphere given by

$$\hat{r} = \begin{bmatrix} \cos\delta\cos\alpha \\ \cos\delta\sin\alpha \\ \sin\delta \end{bmatrix}. \tag{1}$$

This vector can be considered as one of a set of three orthogonal unit vectors at the point $\hat{r}$ on the unit sphere, the other two being given by

$$\hat{\alpha} = \begin{bmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{bmatrix}, \quad \hat{\delta} = \begin{bmatrix} -\sin\delta\cos\alpha \\ -\sin\delta\sin\alpha \\ \cos\delta \end{bmatrix} \tag{2}$$

where $\hat{\alpha}$ and $\hat{\delta}$ are respectively in the directions of locally increasing right ascension and declination, as can be seen be differentiating $\hat{r}$ with respect to α or δ and then renormalizing. Clearly, $\hat{\alpha}$ and $\hat{\delta}$ lie in the plane tangential to the unit sphere at the point $\hat{r}$, and normal to the direction of the vector $\hat{r}$.

1.2 The equation of a spiral

The equation of a spiral in the plane in polar coordinates is

$$r = \frac{w}{2\pi}\theta \tag{3}$$

where w is the width between two successive passes through equivalent angles. For any vector expressed in polar coordinates, the speed is given by

$$s = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}. \tag{4}$$

We wish to traverse our spiral at a constant speed. Substituting 3 into 4 one finds

$$s = \frac{w}{2\pi}\dot{\theta}\sqrt{1+\theta^2} \tag{5}$$

and solving for $\frac{d\theta}{dt}$ gives

$$\frac{d\theta}{dt} = \frac{2\pi s}{w\sqrt{1+\theta^2}}. \tag{6}$$

In principle, this differential equation can be solved by separation of variables,[2] but in practice it is unnecessary to do so since for our purposes the smallest

[1] With some abuse of notation the symbol $\hat{r}$ will be used to refer to both a point on the surface of the unit sphere and to the vector which connects the origin to that point.

[2] For those who are dying to know, the solution is

$$\frac{\theta\sqrt{1+\theta^2}}{2} + \frac{\sinh^{-1}\theta}{2} = \frac{2\pi s}{w}t.$$

Solving for $\theta(t)$ is left as an exercise for the reader.

time interval of interest is the period of one master card interrupt, Δt. In effect, we integrate this differential equation numerically:

$$\begin{aligned} \Delta\theta(t) &= \frac{2\pi s}{w\sqrt{1+\theta(t)^2}}\Delta t \\ \theta(t+\Delta t) &= \theta(t) + \Delta\theta(t). \end{aligned} \tag{7}$$

Thus, given θ from the previous interrupt and the time interval between interrupts we can calculate θ for the current interrupt. Once we have θ, we can get r from 3, and then convert to rectangular coordinates as

$$\begin{aligned} x(t) &= \frac{w}{2\pi}\theta(t)\cos\theta(t) \\ y(t) &= \frac{w}{2\pi}\theta(t)\sin\theta(t). \end{aligned} \tag{8}$$

where $x(t)$ and $y(t)$ are the cartesian coordinates corresponding to polar coordinates $r(t)$ (as given in 3) and $\theta(t)$ (given by the solution of 6).

1.3 Spiral on the Celestial Sphere

Our goal is to execute a spiral search on the plane locally tangent to the nominal position of the star on the celestial sphere. Combining 6 and 8 gives us the cartesian coordinates $x(t)$ and $y(t)$ of a constant speed spiral trajectory in a plane. Equation 2 gives two vectors $\hat{\alpha}$ and $\hat{\delta}$ which define the plane tangent to the nominal position of the star and therefore

$$\hat{r}'(t) = \hat{r} + x(t)\hat{\alpha} + y(t)\hat{\delta} \tag{9}$$

is the equation of the spiral we want.

Notice that since $\hat{\alpha}$ and $\hat{\delta}$ are perpendicular to $\hat{r}$, $\hat{r}'$ is appoximately a unit vector for small x and y. However, since $\hat{r}'$ remains in the plane tanget to the unit sphere at $\hat{r}$ and proper direction vectors lie in the unit sphere, it is in general necessary to renormalize $\hat{r}'$.