1 The Pointing Model

The pointing model is the mathematical transformation which converts a star's apparent topocentric position to the roll and tilt positions necessary to point the siderostat. The transformation consists of three invertible steps, each of which introduces two parameters, for a total of six.

1.1 Pedestal orientation

The two parameters which describe the relationship of the telescope coordinate system to the topocentric coordinate system are the azimuth A and altitude a of the roll (z) axis. Astronomical convention measures azimuths positive from north eastward, and therefore to move the z axis of the topocentric coordinate system to a given azimuth A requires a rotation by $-A$ around the x axis. It can then be moved to the altitude a by a rotation by a around the y axis. The two rotations in succession are described by the matrix

$$\begin{bmatrix} \cos a & 0 & -\sin a \\ 0 & 1 & 0 \\ \sin a & 0 & \cos a \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos A & -\sin A \\ 0 & \sin A & \cos A \end{bmatrix} = \begin{bmatrix} \cos a & -\sin a \sin A & -\sin a \cos A \\ 0 & \cos A & -\sin A \\ \sin a & \cos a \sin A & \cos a \cos A \end{bmatrix} \tag{1}$$

The matrix which describes the transformation from telescope to local topocentric coordinates is

$$\mathsf{M} = \begin{bmatrix} \cos D & 0 & -\sin D \\ \sin D \sin A & \cos A & \cos D \sin A \\ \sin D \cos A & -\sin A & \cos D \cos A \end{bmatrix} \tag{2}$$

Since this is a unitary transformation, the inverse of this matrix is its transpose and defines the transformation from local topocentric to telescope coordinates. Therefore, if $\hat{\mathsf{s}}$ is the star's position unit vector in local topocentric coordinates, then its position vector expressed in telescope coordinates $\hat{\mathsf{x}}$ is

$$\hat{\mathsf{x}} = \mathsf{M}^{\mathrm{T}} \hat{\mathsf{s}} \quad \text{or} \quad x_j = \sum_{i=1}^{3} M_{ij} s_i \tag{3}$$

Obviously, this operation can be inverted to get $\hat{\mathsf{s}}$ from $\hat{\mathsf{x}}$

$$\hat{\mathsf{s}} = \mathsf{M} \hat{\mathsf{x}}. \tag{4}$$

1.2 Optical axis

The optical axis of the telescope nominally should coincide with the roll axis of the siderostat, which is the z axis in the telescope coordinate system. In practice, this is not the case, and in order to point the telescope we must place the vector normal to the siderostat so that it bisects the angle between the star's position vector $\hat{\mathsf{x}}$ and the optical axis of the telescope $\hat{\mathsf{v}}$. The angles θ and ϕ which describe the position of the optical axis $\hat{\mathsf{v}}$, i.e.

$$\hat{\mathsf{v}} = \begin{bmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ \sin\theta \end{bmatrix} \tag{5}$$

and which can be thought of as the tilt and roll positions of the optical axis, are the next two parameters in the pointing model. The unit vector $\hat{\mathsf{y}}$ which bisects the angle between $\hat{\mathsf{x}}$ and $\hat{\mathsf{v}}$ is given by

$$\hat{\mathsf{y}} = \frac{\hat{\mathsf{x}} + \hat{\mathsf{v}}}{|\hat{\mathsf{x}} + \hat{\mathsf{v}}|} \quad \text{or} \quad y_j = \frac{x_j + v_j}{|\hat{\mathsf{x}} + \hat{\mathsf{v}}|}. \tag{6}$$

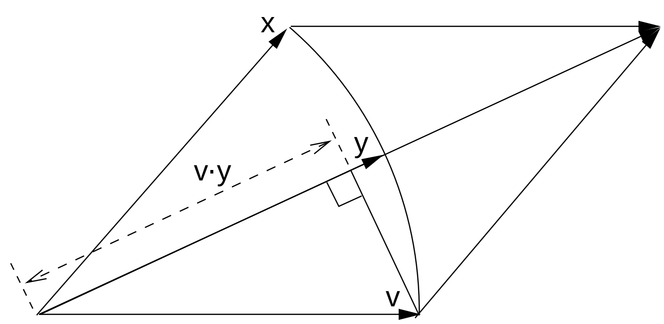

Figure 1: $\hat{\mathsf{y}} = \frac{\hat{\mathsf{x}}+\hat{\mathsf{v}}}{|\hat{\mathsf{x}}+\hat{\mathsf{v}}|}$

This operation can be inverted to get $\hat{\mathsf{x}}$ given $\hat{\mathsf{y}}$ and $\hat{\mathsf{v}}$. As can be seen from Figure (1), the length of the vector $\hat{\mathsf{x}}+\hat{\mathsf{v}}$ is twice $\hat{\mathsf{v}}\cdot\hat{\mathsf{y}}$, and since $\hat{\mathsf{y}}$ is a unit vector in the direction $\hat{\mathsf{x}}+\hat{\mathsf{v}}$, $\hat{\mathsf{x}}+\hat{\mathsf{v}} = 2(\hat{\mathsf{v}}\cdot\hat{\mathsf{y}})\hat{\mathsf{y}}$. Therefore

$$\hat{\mathsf{x}} = 2(\hat{\mathsf{v}}\cdot\hat{\mathsf{y}})\hat{\mathsf{y}} - \hat{\mathsf{v}} \tag{7}$$

1.3 Home positions

The home positions of the roll and tilt axes are defined by opto-interrupters which have been placed on the axes near the zero positions defined by the telescope coordinate system. Since it is not practical to try to place the opto-interrupters exactly on the zero positions, and yet it is convenient to define the home positions to be the zero positions, two additive constants R_0 and T_0 are introduced to account for the home positions of the axes. These are the last two parameters in the pointing model. Using equation ??, the roll and tilt positions to point the normal to the siderostat in the direction $\hat{\mathsf{y}}$ are

$$\begin{aligned} R &= \arctan(\frac{y_2}{y_1}) + R_0 \\ T &= \arcsin(y_3) + T_0. \end{aligned} \tag{8}$$

A Partial derivatives

This appendix gives the derivations of the partial derivatives of the roll and tilt positions with respect to the pointing parameters, which are used to build the normal equations for fitting the pointing model.

A.1 Preliminaries

Notice that if p_α is any of the parameters D, A, θ, or ϕ, then

$$\frac{\partial R}{\partial p_\alpha} = \frac{y_1^2}{y_1^2 + y_2^2} \frac{y_1 \frac{\partial y_2}{\partial p_\alpha} - y_2 \frac{\partial y_1}{\partial p_\alpha}}{y_1^2} = \frac{y_1 \frac{\partial y_2}{\partial p_\alpha} - y_2 \frac{\partial y_1}{\partial p_\alpha}}{y_1^2 + y_2^2} \tag{9}$$

and

$$\frac{\partial T}{\partial p_\alpha} = \frac{1}{\sqrt{1 - y_3^2}} \frac{\partial y_3}{\partial p_\alpha}. \tag{10}$$

Also notice that since

$$y_k = \frac{x_k + v_k}{|\hat{\mathsf{x}} + \hat{\mathsf{v}}|}$$

the dependence on D and A enters only through $\hat{\mathsf{x}}$ and the dependence on θ and ϕ enters only through $\hat{\mathsf{v}}$. Therefore, if p_β is either D or A

$$\frac{\partial y_k}{\partial p_\beta} = \frac{|\hat{\mathsf{x}} + \hat{\mathsf{v}}| \frac{\partial x_k}{\partial p_\beta} - (x_k + v_k) \frac{\partial}{\partial p_\beta} |\hat{\mathsf{x}} + \hat{\mathsf{v}}|}{|\hat{\mathsf{x}} + \hat{\mathsf{v}}|^2} = \frac{1}{|\hat{\mathsf{x}} + \hat{\mathsf{v}}|} \left(\frac{\partial x_k}{\partial p_\beta} - y_k \frac{\partial}{\partial p_\beta} |\hat{\mathsf{x}} + \hat{\mathsf{v}}| \right) \tag{11}$$

and if p_γ is either θ or ϕ

$$\frac{\partial y_k}{\partial p_\gamma} = \frac{1}{|\hat{\mathsf{x}} + \hat{\mathsf{v}}|} \left(\frac{\partial v_k}{\partial p_\gamma} - y_k \frac{\partial}{\partial p_\gamma} |\hat{\mathsf{x}} + \hat{\mathsf{v}}| \right). \tag{12}$$

If we insert equation (11) into equation (9) and take advantage of the fact that

$$\begin{aligned} y_1 \frac{\partial y_2}{\partial p_\beta} - y_2 \frac{\partial y_1}{\partial p_\beta} &= \frac{y_1}{|\hat{\mathsf{x}} + \hat{\mathsf{v}}|} \left(\frac{\partial x_2}{\partial p_\beta} - y_2 \frac{\partial}{\partial p_\beta} |\hat{\mathsf{x}} + \hat{\mathsf{v}}| \right) \\ &\quad - \frac{y_2}{|\hat{\mathsf{x}} + \hat{\mathsf{v}}|} \left(\frac{\partial x_1}{\partial p_\beta} - y_1 \frac{\partial}{\partial p_\beta} |\hat{\mathsf{x}} + \hat{\mathsf{v}}| \right) \\ &= \frac{1}{|\hat{\mathsf{x}} + \hat{\mathsf{v}}|} \left(y_1 \frac{\partial x_2}{\partial p_\beta} - y_2 \frac{\partial x_1}{\partial p_\beta} \right) \end{aligned}$$

we find that

$$\frac{\partial R}{\partial p_\beta} = \frac{1}{(y_1^2+y_2^2)|\hat{\mathsf{x}}+\hat{\mathsf{v}}|}\left(y_1\frac{\partial x_2}{\partial p_\beta} - y_2\frac{\partial x_1}{\partial p_\beta}\right) \tag{13}$$

and similarly if we insert equation (12) into equation (9)

$$\frac{\partial R}{\partial p_\gamma} = \frac{1}{(y_1^2+y_2^2)|\hat{\mathsf{x}}+\hat{\mathsf{v}}|}\left(y_1\frac{\partial v_2}{\partial p_\gamma} - y_2\frac{\partial v_1}{\partial p_\gamma}\right). \tag{14}$$

Equations (11) and (10) give us

$$\frac{\partial T}{\partial p_\beta} = \frac{1}{\sqrt{1-y_3^2}|\hat{\mathsf{x}}+\hat{\mathsf{v}}|}\left(\frac{\partial x_3}{\partial p_\beta} - y_3\frac{\partial}{\partial p_\beta}|\hat{\mathsf{x}}+\hat{\mathsf{v}}|\right) \tag{15}$$

and equations (12) and (10) give

$$\frac{\partial T}{\partial p_\gamma} = \frac{1}{\sqrt{1-y_3^2}|\hat{\mathsf{x}}+\hat{\mathsf{v}}|}\left(\frac{\partial v_3}{\partial p_\gamma} - y_3\frac{\partial}{\partial p_\gamma}|\hat{\mathsf{x}}+\hat{\mathsf{v}}|\right). \tag{16}$$

A.2 Subderivatives

Inspection of equations (13) through (16) reveals that we must compute $\frac{\partial x_k}{\partial p_\beta}$, $\frac{\partial v_k}{\partial p_\gamma}$, and $\frac{\partial}{\partial p_\alpha}|\hat{\mathsf{x}}+\hat{\mathsf{v}}|$. To compute $\frac{\partial x_k}{\partial p_\beta}$, notice that

$$\begin{array}{ll}
\frac{\partial M_{i1}}{\partial D} = M_{i3} & \frac{\partial M_{i1}}{\partial A} = M_{i2}\sin D \\
\frac{\partial M_{i2}}{\partial D} = 0 & \frac{\partial M_{i2}}{\partial A} = -M_{i1}\sin D - M_{i3}\cos D \\
\frac{\partial M_{i3}}{\partial D} = -M_{i1} & \frac{\partial M_{i3}}{\partial A} = M_{i2}\cos D
\end{array}$$

and therefore since $\hat{\mathsf{s}}$ is independent of all pointing parameters and

$$x_k = \sum_{i=1}^{3} M_{ik} s_i$$

we have

$$\begin{array}{ll}
\frac{\partial x_1}{\partial D} = x_3 & \frac{\partial x_1}{\partial A} = x_2\sin D \\
\frac{\partial x_2}{\partial D} = 0 & \frac{\partial x_2}{\partial A} = -x_1\sin D - x_3\cos D \\
\frac{\partial x_3}{\partial D} = -x_1 & \frac{\partial x_3}{\partial A} = x_2\cos D
\end{array} \tag{17}$$

From the definition of $\hat{\mathsf{v}}$, equation (5) one finds

$$\begin{array}{ll}
\frac{\partial v_1}{\partial \theta} = -v_3\cos\phi & \frac{\partial v_1}{\partial \phi} = -v_2 \\
\frac{\partial v_2}{\partial \theta} = -v_3\sin\phi & \frac{\partial v_2}{\partial \phi} = v_1 \\
\frac{\partial v_3}{\partial \theta} = v_1\cos\phi + v_2\sin\phi & \frac{\partial v_3}{\partial \phi} = 0
\end{array} \tag{18}$$

To calculate the derivatives of $|\hat{\mathsf{x}}+\hat{\mathsf{v}}|$, first notice that

$$\frac{\partial}{\partial p_\beta}|\hat{\mathsf{x}}+\hat{\mathsf{v}}| = \frac{\partial}{\partial p_\beta}\sqrt{\sum_{i=1}^{3}(x_i+v_i)^2} = \frac{\sum_{i=1}^{3}(x_i+v_i)\frac{\partial x_i}{\partial p_\beta}}{|\hat{\mathsf{x}}+\hat{\mathsf{v}}|} \tag{19}$$

and similarly

$$\frac{\partial}{\partial p_\gamma}|\hat{\mathsf{x}}+\hat{\mathsf{v}}| = \frac{\sum_{i=1}^{3}(x_i+v_i)\frac{\partial v_i}{\partial p_\gamma}}{|\hat{\mathsf{x}}+\hat{\mathsf{v}}|} \tag{20}$$

Expanding the sum in (19) and using (17) we find

$$\begin{aligned}\frac{\partial}{\partial D}|\hat{\mathsf{x}}+\hat{\mathsf{v}}| &= \frac{(x_1+v_1)x_3-(x_3+v_3)x_1}{|\hat{\mathsf{x}}+\hat{\mathsf{v}}|}\\ &= \frac{v_1x_3-v_3x_1}{|\hat{\mathsf{x}}+\hat{\mathsf{v}}|}\end{aligned} \tag{21}$$

and

$$\begin{aligned}\frac{\partial}{\partial A}|\hat{\mathsf{x}}+\hat{\mathsf{v}}| &= \frac{1}{|\hat{\mathsf{x}}+\hat{\mathsf{v}}|}[(x_1+v_1)x_2\sin D-(x_2+v_2)(x_1\sin D+x_3\cos D)\\ &\quad +(x_3+v_3)x_2\cos D]\\ &= \frac{1}{|\hat{\mathsf{x}}+\hat{\mathsf{v}}|}[(v_1x_2-v_2x_1)\sin D-(v_2x_3-v_3x_2)\cos D]\end{aligned} \tag{22}$$

Expanding the sum in (20) and using (18) we find

$$\begin{aligned}\frac{\partial}{\partial \theta}|\hat{\mathsf{x}}+\hat{\mathsf{v}}| &= \frac{1}{|\hat{\mathsf{x}}+\hat{\mathsf{v}}|}[-(x_1+v_1)v_3\cos\phi-(x_2+v_2)v_3\sin\phi\\ &\quad +(x_3+v_3)(v_1\cos\phi+v_2\sin\phi)]\\ &= \frac{1}{|\hat{\mathsf{x}}+\hat{\mathsf{v}}|}[(x_3v_1-x_1v_3)\cos\phi+(x_3v_2-x_2v_3)\sin\phi]\end{aligned} \tag{23}$$

and

$$\begin{aligned}\frac{\partial}{\partial \phi}|\hat{\mathsf{x}}+\hat{\mathsf{v}}| &= \frac{-(x_1+v_1)v_2+(x_2+v_2)v_1}{|\hat{\mathsf{x}}+\hat{\mathsf{v}}|}\\ &= \frac{x_2v_1-x_1v_2}{|\hat{\mathsf{x}}+\hat{\mathsf{v}}|}\end{aligned} \tag{24}$$

A.3 The Punchline

Combining equations (13) and (17) we have

$$\frac{\partial R}{\partial D} = \frac{-y_2x_3}{(y_1^2+y_2^2)|\hat{\mathsf{x}}+\hat{\mathsf{v}}|} \tag{25}$$

and

$$\frac{\partial R}{\partial A} = \frac{-(x_1y_1+x_2y_2)\sin D-x_3y_1\cos D}{(y_1^2+y_2^2)|\hat{\mathsf{x}}+\hat{\mathsf{v}}|}. \tag{26}$$

Combining equations (14) and (18) we have

$$\frac{\partial R}{\partial \theta} = \frac{y_2 v_3 \cos\phi - y_1 v_3 \sin\phi}{(y_1^2 + y_2^2)|\hat{\mathsf{x}} + \hat{\mathsf{v}}|} \tag{27}$$

and

$$\frac{\partial R}{\partial \phi} = \frac{y_1 v_1 + y_2 v_2}{(y_1^2 + y_2^2)|\hat{\mathsf{x}} + \hat{\mathsf{v}}|}. \tag{28}$$

And of course, trivially

$$\frac{\partial R}{\partial R_0} = 1 \qquad \text{and} \qquad \frac{\partial R}{\partial T_0} = 0. \tag{29}$$

Combining equations (15), (17), and (21) we have

$$\frac{\partial T}{\partial D} = \frac{-1}{\sqrt{1 - y_3^2}|\hat{\mathsf{x}} + \hat{\mathsf{v}}|}(x_1 + y_3 \frac{v_1 x_3 - v_3 x_1}{|\hat{\mathsf{x}} + \hat{\mathsf{v}}|}) \tag{30}$$

and using (15), (17), and (22)

$$\frac{\partial T}{\partial A} = \frac{1}{\sqrt{1 - y_3^2}|\hat{\mathsf{x}} + \hat{\mathsf{v}}|}(x_2 \cos D - y_3 \frac{(v_1 x_2 - v_2 x_1)\sin D - (v_2 x_3 - v_3 x_2)\cos D}{|\hat{\mathsf{x}} + \hat{\mathsf{v}}|}). \tag{31}$$

Combining equations (16), (18), and (23) we have

$$\begin{aligned} \frac{\partial T}{\partial \theta} &= \frac{1}{\sqrt{1 - y_3^2}|\hat{\mathsf{x}} + \hat{\mathsf{v}}|} \times \\ & (v_1 \cos\phi + v_2 \sin\phi - y_3 \frac{(x_3 v_1 - x_1 v_3)\cos\phi + (x_3 v_2 - x_2 v_3)\sin\phi}{|\hat{\mathsf{x}} + \hat{\mathsf{v}}|}) \end{aligned} \tag{32}$$

and using (16), (18), and (24)

$$\frac{\partial T}{\partial \phi} = \frac{-y_3(x_2 v_1 - x_1 v_2)}{\sqrt{1 - y_3^2}|\hat{\mathsf{x}} + \hat{\mathsf{v}}|^2}. \tag{33}$$

And, finally, it is trivial that

$$\frac{\partial T}{\partial R_0} = 0 \qquad \text{and} \qquad \frac{\partial T}{\partial T_0} = 1 \tag{34}$$

Equations (25) through (34) give the partial derivatives of the observables (roll and tilt positions where the stars are found) with respect to the pointing model parameters (D, A, θ, ϕ, V_0, and S_0).

A.4 Implementation

The following function evaluates the partial dervatives for the pointing model passed in **this** at the topocentric vector passed in topo and returns the values in the arrays dR and dT. The elements of the array dR

```
void partials(pointing_model *this, double topo[ ], double dR[ ], double
dT[ ])
{
  double x[3], y[3], v[3], m, f, sD, cD, sP, cP;

  vector_copy(v, this→v);
  vector_multiply(x, this→MT, topo);
  vector_sum(y, x, v);
  m = modulus(y);
  normalize(y);
  cD=cos(this→dip);
  sD=sin(this→dip);
  cP=cos(this→phi);
  sP=sin(this→phi);
  f = 1/((y[0]*y[0]+y[1]*y[1])*m);
  dR[0] = -f*y[1]*x[2];
  dR[1] = -f*((x[0]*y[0]+x[1]*y[1])*sD+x[2]*y[0]*cD);
  dR[2] = f*(y[1]*v[2]*cP-y[0]*v[2]*sP);
  dR[3] = f*(y[0]*v[0]+y[1]*v[1]);
  dR[4] = 1;
  dR[5] = 0;
  f = 1/(sqrt(1-y[2]*y[2])*m);
  dT[0] = -f*(x[0]+y[2]*(v[0]*x[2]-v[2]*x[0])/m);
  dT[1] =
    f*(x[1]*cD-y[2]*((v[0]*x[1]-v[1]*x[0])*sD-(v[1]*x[2]-v[2]*x[1])*cD)/m);
  dT[2] = f*(v[0]*cP+v[1]*sP-
           y[2]*((x[2]*v[0]-x[0]*v[2])*cP+(x[2]*v[1]-x[1]*v[2])*sP)/m);
  dT[3] = -f*y[2]*(x[1]*v[0]-x[0]*v[1])/m;
  dT[4] = 0;
  dT[5] = 1;
}
```